On Uniqueness of Kerr Space-time near null infinity

Xiaoning Wu[a1] and Shan Bai[b2]

a. Institute of Mathematics,
Academy of Mathematics and Systems Science,
Chinese Academy of Sciences,
P.O.Box 2734, Beijing, 100080, China.

b. Institute of Theoretical Physics,
Chinese Academy of Sciences,
P.O.Box 2735, Beijing, 100080, China.

Abstract

We re-express the Kerr metric in standard Bondi-Saches' coordinate near null infinity $\mathcal{I}^+$. Using the uniqueness result of characteristic initial value problem, we prove the Kerr metric is the only asymptotic flat, stationary, axial symmetric, Type-D solution of vacuum Einstein equation. The Taylor series of Kerr space-time is expressed in terms of B-S coordinates and the N-P constants have been calculated.

keywords: Kerr solution, Uniqueness Theorem

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1 Introduction

After the work by Bondi et.al.[1], it is well-known the Bondi coordinates is a very natural choice when we want to describe the asymptotic behavior of gravitational field near null infinity $\mathcal{I}^+$. Based on works by Penrose, Newman and Unti[2, 3], there is an elegant way to re-express Bondi's work in N-P formulism. This also gives us a general formulism to describe the asymptotic structure of general asymptotic flat space-time which is smooth enough near $\mathcal{I}^+$. Using characteristic initial value (CIV) problem method, many authors[8, 9, 10] have shown the existence of null infinity in general case and pointed out the degree of freedom of gravitational field near null infinity. The CIV method has many advantages in dealing with gravitational radiation problem. Recently, this method has been used in numerical relativity. Winicour and his colleagues develop the CCM method. They want to combine the CIV method and standard Hamiltonian evolution method[11]. On the other hand, Kerr solution is a very important exact solution of Einstein equation both in theoretical area and in application. An interesting question is whether Kerr metric describes the space-time outside a stationary rotating star. For long time, the Bondi coordinates of Kerr space-time is not very clear. For example, how to describe the Kerr space-time in Unti-Newman's general formalism[3]? The uniqueness theorem[12, 13] tells us that Kerr solution is the only asymptotic flat, stationary, axial symmetric solution of vacuum Einstein equation with regular event horizon. From the application point of view, it is very difficult to get the detail

[1]e-mail : wuxn@amss.ac.cn
[2]e-mail : xiaobai99033@yahoo.com.cn

information about the event horizon of a space-time because of the infinite red-shift near horizon. An interesting question is how to identify the Kerr solution based on information near null infinity. This is more practical in future gravitational experiments. This idea can also be understood from the Geroch conjecture[14]. Obviously, stationary and axial symmetric condition is not enough because there are many asymptotic flat exact solutions of Ernst equation. In next section, it is found such uncertainty comes from the homogeneous part of a control equation which comes from the Killing equation. The general solutions of that equation contain some free constants. These unknown constants are close related with Geroch-Hansen multi-pole moments[17]. In order to identify the Kerr solution, we use Petrov classification[4] and show that condition will help us to pick out Kerr solution finally. Further more, N-P constants for Kerr space-time is also calculated as a byproduct.

This paper is organized as following : In section II, we prove an local uniqueness theorem of Kerr solution based on the information near null infinity. This theorem also tells us the standard Unti-Newman expansion of Kerr metric. The detail expression of this extension is contained in Appendix A. Appendix B contains some spin-weight harmonics which is useful for our calculation.

2 Main theorem

Let (M, g) be an asymptotic flat space-time, (u, r, θ, φ) be the standard Bondi-Sachs coordinates.

Theorem 1 *Suppose (M, g) be an asymptotic flat, stationary, axial symmetric, Type-D, vacuum space-time with in a neighborhood of null infinity, then it is isometric to Kerr space-time in the Bondi coordinates neighborhood.*

In order to make the proof clearly enough, we divide it into two subsections. In first subsection, we calculate the Taylor series of general stationary axially symmetric space-time in Bondi coordinates. We find all Taylor coefficients can be expressed in terms of $\{\Psi_0^k\}$ and their derivatives. Unknown functions $\{\Psi_0^k\}$ satisfy a linear inhomogeneous second order partial differential equation. We find the general solution of such equation has the form $\Psi_0^k = \tilde{\Psi}_0^k + D^k\ {}_2Y_{k+2,0}$, where $\tilde{\Psi}_0^k$ is the special solution which corresponds to Kerr metric and D^k is a free constant, i.e. all axially symmetric and stationary solutions are characterized by the set of constant $\{D^k\}$. This set of constants are close related with the famous Geroch-Hansen multipole. In section 2.2, with the help of Type-D condition, we show all $\{D^k\}$ vanish and Kerr metric is the unique stationary, axial-symmetric, asymptotic flat Tpye-D metric .

2.1 Taylor series of general axial symmetric vacuum stationary spacetime

Suppose (M, g) be a vacuum stationary axial symmetric space-time. t^a and ϕ^a are two commutative Killing vectors. Near $\mathcal{I}^+$, we use the standard B-S coordinates to do the standard asymptotic expansion. The detailed construction of these coordinates is well-known and can be found in Re.[2, 3]. With this choice of coordinates, we also can choose a set of

null tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$, such that $l^a = \left(\frac{\partial}{\partial r}\right)^a$ and these tetrad are parallel-transported along l^a. Under such choice of coordinates, $\phi^a = \left(\frac{\partial}{\partial \varphi}\right)^a$. The time-like Killing vector t^a can be expressed in terms of null tetrad as $t^a = Tl^a + Rn^a + A\bar{m}^a + \bar{A}m^a$. $[t^a, \phi^a] = 0$ means that T, R, A are independent of φ. The null tetrad components of Killing equation for t^a are

$$
\begin{aligned}
&-DR + (\varepsilon + \bar{\varepsilon})R + \bar{\kappa}A + \kappa\bar{A} = 0, \\
&-DT - (\varepsilon + \bar{\varepsilon})T - \pi A - \bar{\pi}\bar{A} - D'R + (\gamma + \bar{\gamma})R + \bar{\tau}A + \tau\bar{A} = 0, \\
&-\kappa T + \bar{\pi}R + DA + (\bar{\varepsilon} - \varepsilon)A - \delta R + (\bar{\alpha} + \beta)R + \bar{\rho}A + \sigma\bar{A} = 0, \\
&-D'T - (\gamma + \bar{\gamma})T - \nu A - \bar{\nu}\bar{A} = 0, \\
&-\tau T + \bar{\nu}R + D'A + (\bar{\gamma} - \gamma)A - \delta T - (\bar{\alpha} + \beta)T - \mu A - \bar{\lambda}\bar{A} = 0, \\
&-\sigma T + \bar{\lambda}R + \delta A + (\bar{\alpha} - \beta)A = 0, \\
&-\rho T + \mu R + \delta\bar{A} - (\bar{\alpha} - \beta)\bar{A} - \bar{\rho}T + \bar{\mu}R + \bar{\delta}A - (\alpha - \bar{\beta})A = 0.
\end{aligned}
$$

Using the Bondi Gauge $\kappa = \varepsilon = \pi = 0$, $\rho = \bar{\rho}$, $\tau = \bar{\alpha} + \beta$, we have

$$
\begin{aligned}
&-DR = 0, && (1)\\
&-DT - D'R + (\gamma + \bar{\gamma})R + \bar{\tau}A + \tau\bar{A} = 0, && (2)\\
&DA - \delta R + \tau R + \bar{\rho}A + \sigma\bar{A} = 0, && (3)\\
&-D'T - (\gamma + \bar{\gamma})T - \nu A - \bar{\nu}\bar{A} = 0, && (4)\\
&-\tau T + \bar{\nu}R + D'A + (\bar{\gamma} - \gamma)A - \delta T - \tau T - \mu A - \bar{\lambda}\bar{A} = 0, && (5)\\
&-\sigma T + \bar{\lambda}R + \delta A + (\bar{\alpha} - \beta)A = 0, && (6)\\
&-\rho T + \mu R + \delta\bar{A} - (\bar{\alpha} - \beta)\bar{A} - \bar{\rho}T + \bar{\mu}R + \bar{\delta}A - (\alpha - \bar{\beta})A = 0. && (7)
\end{aligned}
$$

Here we use the standard notation of [2, 3, 4]. Differential operators in above equations are defined as

$$
\begin{aligned}
D &:= \frac{\partial}{\partial r}, \\
D' &:= \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X\frac{\partial}{\partial \zeta} + \bar{X}\frac{\partial}{\partial \bar{\zeta}}, \\
\delta &:= \omega\frac{\partial}{\partial r} + \xi^3\frac{\partial}{\partial \zeta} + \xi^4\frac{\partial}{\partial \bar{\zeta}}, \quad \zeta = e^{i\varphi}\cot\frac{\theta}{2}. \qquad (8)
\end{aligned}
$$

It is well known that stationary solutions to Einstein's vacuum field equations are analytic[5]. Moreover, it is also known that asymptotically flat stationary vacuum solutions are not only analytic, but even admit an analytic conformal extension through null infinity[6, 7]. Keeping this result in mind, all geometric quantities (the coordinate components of null tetrad, N-P coefficients, components of time-like Killing vector and components of Weyl curvature) can be expressed in terms of power series of $\frac{1}{r}$, for example

$$
\begin{aligned}
T &= T^0 + \frac{T^1}{r} + \cdots, \\
A &= A^0 + \frac{A^1}{r} + \cdots\ , \qquad (9)
\end{aligned}
$$

some lower order Taylor coefficients of components of tetrad, N-P coefficients and components of Weyl tensor (up to 3th order) can be found in section 9.8 of [2].

First of all, let's consider the function R. Eq.(1) and axial symmetric condition tell us that $R = R(u, \theta)$. With the formal expansion of null tetrad and N-P coefficients[2, 3], zero order of Eq.(2) gives

$$\frac{\partial R}{\partial u} = 0, \tag{10}$$

so $R = R(\theta, \phi)$. In order to get more information about R, higher order of Killing equation are needed. The first order of Eq.(3), Eq.(5) and Eq.(7) are

$$\begin{aligned} \delta_0 R + A^0 &= 0, & (11)\\ -\bar{\Psi}_3^0 R + \dot{A}^1 - \delta_0 T^0 + \frac{1}{2} A^0 - \dot{\sigma}^0 \bar{A}^0 &= 0, & (12)\\ 2T^0 - R &= 0, & (13) \end{aligned}$$

where $\delta_0 = \frac{(1+\zeta\bar{\zeta})}{\sqrt{2}} \frac{\partial}{\partial \zeta}$ and " $\cdot$ " means ∂_u . Because the space-time is stationary, there is no Bondi flux. This implies $\dot{\sigma}^0 = 0$ [2], then N-P equations tell us this leads $\Psi_3^0 = 0$. Combining this condition with (11),(12) and (13), we get

$$\dot{A}^1 + A^0 = 0. \tag{14}$$

The second order of Eq.(3) is

$$2A^1 = \frac{\zeta(1 + \zeta\bar{\zeta})}{\sqrt{2}\ \bar{\zeta}} \sigma^0 \frac{\partial R}{\partial \zeta} + \sigma^0 \bar{A}^0. \tag{15}$$

It has been shown that the right hand side of above equation is independent on u, so $\dot{A}^1 = 0$. This implies $A^0 = 0$ and R is a constant. So we can take $R = 1$ without lost of generality.

With $R = 1$, Killing equations become

$$\begin{aligned} &-DT + (\gamma + \bar{\gamma}) + \bar{\tau} A + \tau \bar{A} = 0, & (16)\\ &DA + \tau + \bar{\rho} A + \sigma \bar{A} = 0, & (17)\\ &-D'T - (\gamma + \bar{\gamma}) T - \nu A - \bar{\nu} \bar{A} = 0, & (18)\\ &-\tau T + \bar{\nu} + D'A + (\bar{\gamma} - \gamma) A - \delta T - \tau T - \mu A - \bar{\lambda} \bar{A} = 0, & (19)\\ &-\sigma T + \bar{\lambda} + \delta A + (\bar{\alpha} - \beta) A = 0, & (20)\\ &-\rho T + \mu + \delta \bar{A} - (\bar{\alpha} - \beta) \bar{A} - \bar{\rho} T + \bar{\mu} + \bar{\delta} A - (\alpha - \bar{\beta}) A = 0. & (21) \end{aligned}$$

With Eq.(9), the non-trivial zero order Killing equation is

$$\dot{T}^0 = 0. \tag{22}$$

Non-trivial first order Killing equations are

$$\begin{aligned} &-A^0 = 0, & (23)\\ &-\dot{T}^1 + \Psi_3^0 A^0 + \bar{\Psi}_3^0 \bar{A}^0 = 0, & (24)\\ &-\bar{\Psi}_3^0 + \dot{A}^1 + \frac{1}{2} A^0 - \dot{\bar{\sigma}}^0 A^0 = 0, & (25)\\ &\dot{\bar{\sigma}}^0 + \delta_0 A^0 + 2\bar{\alpha}^0 A^0 = 0, & (26)\\ &2T^0 - 1 = 0, & (27) \end{aligned}$$

which implies $\dot{T}^1 = 0$ and $\dot{A}^1 = 0$. Same order N-P equations also give $\Psi_3^0 = 0$ and $\Psi_4^0 = 0$.

Second order Killing equations are

$$
\begin{aligned}
& T^1 - \frac{1}{2}(\Psi_2^0 + \bar{\Psi}_2^0) = 0, && (28)\\
& -2A^1 = 0, && (29)\\
& \dot{T}^2 = 0, && (30)\\
& \frac{1}{2}\eth\Psi_2^0 + \dot{A}^2 - \delta_0 T^1 = 0, && (31)\\
& \frac{1}{2}\sigma^0 = 0, && (32)\\
& 2T^1 - \sigma^0\dot{\bar{\sigma}}^0 - \bar{\sigma}^0\dot{\sigma}^0 - \Psi_2^0 - \bar{\Psi}_2^0 = 0, && (33)
\end{aligned}
$$

where $\eth$ is the spin-weight operator[2] and is defined as $\eth f := (\delta_0 + 2s\bar{\alpha}^0)f$, $\alpha^0 = -\frac{\cot\theta}{2\sqrt{2}}$. From these equations, we know $\sigma^0 = 0$, $A^1 = 0$, $\dot{T}^2 = 0$, $T^1 = \frac{1}{2}(\Psi_2^0 + \bar{\Psi}_2^0)$, $\Psi_2^0 = \bar{\Psi}_2^0$, $\dot{\Psi}_2^0 = 0$, $\dot{A}^2 = -\frac{1}{2}\eth\Psi_2^0 + \frac{1}{2}\delta_0(\Psi_2^0 + \bar{\Psi}_2^0)$. It is worth to point out that the result $\dot{\sigma}^0 = 0$ tells us that the Bondi coordinates which is chosen as [3] are associated with the "good cut" of stationary spacetime, i.e. $\sigma^0 = 0$, so the freedom of super-translation has been removed.

The third order killing equations are

$$
\begin{aligned}
& 2T^2 + \frac{1}{3}(\bar{\eth}\Psi_1^0 + \eth\bar{\Psi}_1^0) = 0, && (34)\\
& -3A^2 - \frac{1}{2}\Psi_1^0 = 0, && (35)\\
& \dot{T}^3 = 0, && (36)\\
& \frac{1}{2}\Psi_1^0 + \bar{\nu}^3 + \dot{A}^3 + \frac{3}{2}A^2 - \delta_0 T^2 = 0, && (37)\\
& \delta_0 A^2 + 2\bar{\alpha}^0 A^2 = 0, && (38)\\
& 2T^2 + \frac{1}{2}\bar{\eth}\Psi_1^0 + \frac{1}{2}\eth\bar{\Psi}_1^0 + \delta_0\bar{A}^2 - 2\bar{\alpha}^0\bar{A}^2 + \bar{\delta}_0 A^2 - 2\alpha^0 A^2 = 0. && (39)
\end{aligned}
$$

Eq.(35),(38) imply

$$
\eth\Psi_1^0 = 0. \qquad (40)
$$

The spin-weight of Ψ_1^0 is 1, so it is a linear combination of spin-weight harmonics $\{{}_1Y_{l,m}\}$. The axial symmetric condition implies $m = 0$. The behavior of spin-weight harmonic under the action of operators $\eth$ and $\bar{\eth}$ are [2]

$$
\begin{aligned}
\eth\ {}_sY_{lm} &= -\sqrt{\frac{(l+s+1)(l-s)}{2}}\ {}_{s+1}Y_{lm},\\
\bar{\eth}\ {}_sY_{lm} &= \sqrt{\frac{(l-s+1)(l+s)}{2}}\ {}_{s-1}Y_{lm},\\
{}_0Y_{lm} &= Y_{lm}. && (41)
\end{aligned}
$$

So we get $\Psi_1^0 = c(u){}_1Y_{1,0} = c(u)\sin\theta$. Detailed calculation on same order N-P equations also give

$$
\nu^3 = -\frac{1}{12}\bar{\Psi}_1^0 - \frac{1}{6}\bar{\eth}^2\Psi_1^0,\ \Psi_3^1 = 0,\ \Psi_3^2 = \frac{1}{2}\bar{\eth}^2\Psi_1^0,\ \Psi_4^1 = \Psi_4^2 = \Psi_4^3 = 0,\ \Psi_4^4 = -\frac{1}{4}\bar{\eth}\Psi_3^3. \qquad (42)
$$

Eq.(34) and $\dot{T}^2 = 0$ gives

$$\dot{c}\bar{\eth}\sin\theta + \dot{\bar{c}}\eth\sin\theta = 2(\dot{c} + \dot{\bar{c}})\cos\theta = 0, \tag{43}$$

which implies $\dot{c} + \dot{\bar{c}} = 0$. The first order Bianchi identities tell us $\dot{\Psi}_1^0 - \eth\Psi_2^0 = 0$, which implies $\Psi_2^0 = -\dot{c}\cos\theta + C_2$ but it is well-known that $\bar{\Psi}_2^0 = \Psi_2^0$ for stationary case[2, 3], so $\dot{c} = 0$. This means $\Psi_1^0 = C_1\sin\theta$ and $\Psi_2^0 = C_2$. The Komar integral shows $-C_2$ is just the Bondi mass of the space-time. Additionally, Eq.(37) tells us that $\dot{A}^3 = 0$.

Forth order Killing equations are

$$3T^3 + (\gamma^4 + \bar{\gamma}^4) = 0, \tag{44}$$

$$4A^3 = \frac{1}{3}\bar{\eth}\Psi_0^0, \tag{45}$$

$$\dot{T}^4 + \frac{1}{3}(\bar{\eth}\Psi_1^0 + \eth\bar{\Psi}_1^0)(\Psi_2^0 + \bar{\Psi}_2^0) = 0, \tag{46}$$

$$\frac{1}{2}\Psi_1^0 T^1 - \frac{1}{3}\bar{\eth}\Psi_0^0 + \bar{\nu}^4 + \dot{A}^4 + (\Psi_2^0 + \bar{\Psi}_2^0)A^2 + \frac{3}{2}A^3 - \delta_0 T^3 + \Psi_2^0 A^2 + \frac{1}{2}A^3 = 0, \tag{47}$$

$$\frac{1}{4}\Psi_0^0 + \bar{\lambda}^4 + \eth A^3 = 0, \tag{48}$$

$$2T^3 + \mu^4 + \bar{\mu}^4 + \eth\bar{A}^3 + \bar{\eth}A^3 = 0. \tag{49}$$

where $\gamma^4 = -\frac{1}{12}(\alpha^0\bar{\eth}\Psi_0^0 - \bar{\alpha}^0\eth\bar{\Psi}_0^0) - \frac{1}{8}\bar{\eth}^2\Psi_0^0$, $\lambda^4 = -\frac{1}{12}\bar{\Psi}_0^0$, $\mu^4 = -\frac{1}{3}\Psi_2^2 = -\frac{1}{6}\bar{\eth}^2\Psi_0^0$, $\nu^4 = \frac{1}{24}(\eth\bar{\Psi}_0^0 + \bar{\eth}^3\Psi_0^0)$, $\Psi_3^3 = -\frac{1}{2}\bar{\Psi}_1^0\Psi_2^0 - \frac{1}{6}\bar{\eth}^3\Psi_0^0$ (These results are got from same order N-P equations).

The spin-weight of Ψ_0 is 2, so Eq.(45),(48) imply

$$\Psi_0^0 = D^5(u)\sin^2\theta, \tag{50}$$

Eq.(37),(45) eliminate time dependence of $D^5(u)$, i.e. $\Psi_0^0 = D^5\sin^2\theta$. Eq.(47) is

$$\dot{A}^4 = 0 \tag{51}$$

Fifth order Killing equations are

$$4T^4 + (\gamma^5 + \bar{\gamma}^5) - \frac{1}{2}\bar{\Psi}_1^0 A^2 - \frac{1}{2}\Psi_1^0\bar{A}^2 = 0, \tag{52}$$

$$A^4 = \frac{1}{5}\tau^5, \tag{53}$$

$$-\frac{1}{2}\sigma^5 + \bar{\lambda}^5 + \frac{1}{2}\Psi_0^0 T^1 + \frac{3}{2}\Psi_1^0 A^2 + \eth A^4 = 0, \tag{54}$$

$$-2\rho^5 + 2T^4 + (\mu^5 + \bar{\mu}^5) + \frac{3}{2}\Psi_1^0\bar{A}^2 + \frac{3}{2}\bar{\Psi}_1^0 A^2 + \eth\bar{A}^4 + \bar{\eth}A^4 = 0, \tag{55}$$

where

$$\begin{aligned}
\rho^5 &= 0,\\
\mu^5 &= -\frac{1}{4}\Psi_2^3,\\
\sigma^5 &= -\frac{1}{3}\Psi_0^1,
\end{aligned}$$

$$
\begin{aligned}
\lambda^5 &= -\frac{1}{8}\bar{\Psi}_0^0\Psi_2^0 - \frac{1}{24}\bar{\Psi}_0^1, \\
\gamma^5 &= -\frac{1}{40}(\alpha^0\bar{\eth}\Psi_0^1 - \bar{\alpha}^0\eth\bar{\Psi}_0^1) + \frac{1}{12}|\Psi_1^0|^2 - \frac{1}{30}\bar{\eth}^2\Psi_0^1, \\
\tau^5 &= \frac{1}{8}\bar{\eth}\Psi_0^1, \\
\Psi_2^3 &= -\frac{2}{3}|\Psi_1^0|^2 + \frac{1}{6}\bar{\eth}^2\Psi_0^1, \\
\Psi_1^2 &= -\frac{1}{2}\bar{\eth}\Psi_0^1.
\end{aligned}
$$

Eq.(52), (46) and Bianchi identities imply $\Psi_1^0 = iC_1\sin\theta,\ C_1 \in \mathbf{R}$, where C_1 is the Komar angular momentum. Eq.(53),(54) give

$$\eth\bar{\eth}\Psi_0^1 + 5\Psi_0^1 = 10(\Psi_1^0)^2 - 15\Psi_0^0\Psi_2^0. \tag{56}$$

The homogeneous part of above equation is

$$\eth\bar{\eth}\Psi_0^1 + 5\Psi_0^1 = 0. \tag{57}$$

Because spin-weight of Ψ_0^1 is 2, it is a linear combination of $\{{}_2Y_{l,0}\}$. Using Eq.(41), the homogeneous equation is

$$(-l^2 - l + 12)\ {}_2Y_{l,0} = 0, \tag{58}$$

which gives $l = 3$. The general solution of Eq.(54) is

$$\Psi_0^1 = \left(\frac{10}{3}(C_1)^2 - 5C_2D^5\right)\sin^2\theta + D^6\ {}_2Y_{3,0}. \tag{59}$$

(Bianchi identities insures $\dot{\Psi}_0^1 = 0$.) By definition[2], the non-zero N-P constant for stationary axial symmetric space-time is

$$G_0 = \frac{10}{3}(C_1)^2 - 5C_2D^5. \tag{60}$$

Until now, we have got series expression of tetrad components up to 4th order, N-P coefficients up to 5th order and Weyl components up to 6th order. To prove this theorem, all Taylor coefficients of all geometric quantities are needed. We use inductive method to solve this problem order by order.

Suppose we have known Taylor coefficients of tetrad components up to $(k-3)^{th}$ order, Taylor coefficients of connections up to $(k-2)^{th}$ order and Taylor coefficients of Weyl curvature components up to $(k-1)^{th}$ order. The $(k-1)^{th}$ order of Killing equation (17) and (20) are

$$-(k-1)A^{k-2} + \tau^{k-1} = \cdots, \tag{61}$$
$$\cdots + \bar{\lambda}^{k-1} + \eth A^{k-2} = 0. \tag{62}$$

where "$\cdots$" means terms which only contain lower order coefficients. Based on the induction hypothesis, those terms are known. In order to solve these equations, we need coefficients

λ^{k-1} and τ^{k-1}. From N-P equations,

$$\begin{aligned}
D\Psi_1 - \bar{\delta}\Psi_0 = -4\alpha\Psi_0 + 4\rho\Psi_1 \quad &\Rightarrow \quad \Psi_1^k = -\frac{1}{(k-4)}\bar{\eth}\Psi_0^k + \cdots . \\
D\sigma = 2\rho\sigma + \Psi_0 \quad &\Rightarrow \quad \sigma^{k-1} = -\frac{1}{(k-3)}\Psi_0^k + \cdots , \\
D\lambda = \rho\lambda + \bar{\sigma}\mu \quad &\Rightarrow \quad \lambda^{k-1} = \frac{1}{2(k-2)}\ \bar{\sigma}^{k-1} + \cdots , \\
D\tau = \tau\rho + \bar{\tau}\sigma + \Psi_1 \quad &\Rightarrow \quad \tau^{k-1} = -\frac{1}{(k-2)}\Psi_1^k + \cdots ,
\end{aligned} \tag{63}$$

Combining Eq.(61), (62) and (63), we get

$$\eth\bar{\eth}\Psi_0^k + \frac{(k+4)(k+1)}{2}\Psi_0^k = \cdots . \tag{64}$$

The homogeneous part of above equation is

$$\eth\bar{\eth}\hat{\Psi}_0^k + \frac{(k+4)(k+1)}{2}\hat{\Psi}_0^k = 0. \tag{65}$$

Because of Eq.(41) and axial symmetric condition, the general solution should be

$$\Psi_0^k = \tilde{\Psi}_0^k + D^k \ {}_2Y_{k+2,0} \ , \tag{66}$$

where $\tilde{\Psi}^k$ is a special solution of eq.(64) and D^k is a constant. Obviously, Kerr solution satisfies all conditions of our theorem, so $\tilde{\Psi}_0^k$ must exist. The concrete form of $\tilde{\Psi}_0^k$ also can be got by direct calculation. One can express the "$\cdots$" terms in Eq.(64) as a linear combination of spin-weight harmonics $\{{}_2Y_{l,0}\}$. The inductive method insures the maximal value of l in that expression will be finite for any given order, then we can get $\tilde{\Psi}_0^k$ by comparing coefficients between bother sides of this equation. With the general solution of Ψ_0^k, Eq.(63) will give τ^{k-1}, σ^{k-1}, λ^{k-1} and Ψ_1^k. Further more, Cartan structure equations and Bianchi equations will help us to get other coefficients,

$$\begin{aligned}
D\rho = \rho^2 + |\sigma|^2 \quad &\Rightarrow \quad -(k-3)\rho^{k-1} = \cdots , \\
D\alpha = \alpha\rho + \beta\bar{\sigma} \quad &\Rightarrow \quad -(k-2)\alpha^{k-1} = \cdots , \\
D\beta = \beta\rho + \alpha\sigma + \Psi_1 \quad &\Rightarrow \quad -(k-2)\beta^{k-1} = \Psi_1^k + \cdots , \\
D\Psi_2 - \bar{\delta}\Psi_1 = 3\rho\Psi_2 - 2\alpha\Psi_1 - \lambda\Psi_0 \quad &\Rightarrow \quad -(k-3)\Psi_2^k = \bar{\eth}\Psi_1^k + \cdots , \\
D\Psi_3 - \bar{\delta}\Psi_2 = 2\rho\Psi_3 - 2\lambda\Psi_1 \quad &\Rightarrow \quad -(k-2)\Psi_3^k = \bar{\eth}\Psi_2^k + \cdots , \\
D\Psi_4 - \bar{\delta}\Psi_3 = \rho\Psi_4 + 2\alpha\Psi_3 - 3\lambda\Psi_2 \quad &\Rightarrow \quad -(k-1)\Psi_4^k = \bar{\eth}\Psi_3^k + \cdots , \\
D\gamma = \tau\alpha + \bar{\tau}\beta + \Psi_2 \quad &\Rightarrow \quad -(k-1)\gamma^{k-1} = \Psi_2^k + \alpha_0\tau^{k-1} - \bar{\alpha}_0\bar{\tau}^{k-1} + \cdots , \\
D\mu = \mu\rho + \lambda\sigma + \Psi_2 \quad &\Rightarrow \quad -(k-2)\mu^{k-1} = \frac{1}{2}\rho^{k-1} + \Psi_2^k + \cdots , \\
D\nu = \tau\lambda + \bar{\tau}\mu + \Psi_3 \quad &\Rightarrow \quad -(k-1)\nu^{k-1} = \frac{1}{2}\bar{\tau}^{k-1} + \Psi_3^k + \cdots , \\
D\xi^3 = \rho\xi^3 + \sigma\bar{\xi}^4 \quad &\Rightarrow \quad -(k-3)\xi^3_{k-2} = \cdots , \\
D\xi^4 = \rho\xi^4 + \sigma\bar{\xi}^3 \quad &\Rightarrow \quad -(k-3)\xi^4_{k-2} = \cdots , \\
D\omega = \rho\omega + \sigma\omega - (\bar{\alpha} + \beta) \quad &\Rightarrow \quad -(k-3)\omega^{k-2} = -\bar{\alpha}^{k-1} - \beta^{k-1} + \cdots , \\
DX = (\bar{\alpha} + \beta)\xi^3 + (\alpha + \bar{\beta})\bar{\xi}^4 \quad &\Rightarrow \quad -(k-2)X^{k-2} = \frac{1 + |\zeta|^2}{\sqrt{2}}(\alpha^{k-1} + \bar{\beta}^{k-1}) + \cdots , \\
DU = (\bar{\alpha} + \beta)\bar{\omega} + (\alpha + \bar{\beta})\omega - \gamma - \bar{\gamma} \quad &\Rightarrow \quad -(k-2)U^{k-2} = -\gamma^{k-1} - \bar{\gamma}^{k-1} + \cdots .
\end{aligned} \tag{67}$$

From above results, we find we can express all $(k-2)^{th}$ order coefficients of tetrad components, $(k-1)^{th}$ order coefficients of connection components and k^{th} order coefficients of Weyl curvature in terms of Ψ_0^k, $\bar{\eth}$ derivatives of Ψ_0^k and lower order coefficients which we have known. The form of Ψ_0^k is given in Eq.(66). This means we can get all those coefficients for any given order.

2.2 Uniqueness of Kerr solution

In above subsection, we have got Taylor series of a general stationary axial symmetric metric. From Eq.(66), we can see that the freedom in each order Taylor coefficients are just the constant D^k ($k \geq 5$). These arbitrary constants should be closely related to the famous Geroch-Hansen multi-pole moments[14, 15, 16, 17]. What we want to do in this section is to pick out the Kerr solution from those possible solutions, i.e. we need to fix value of $\{D^k\}$. In order to do that, we consider the Petrov classification[4]. It is well known that the Kerr solution belongs to Type-D class, i.e. its Weyl curvature satisfies[4]

$$K = \Psi_1(\Psi_4)^2 - 3\Psi_4\Psi_3\Psi_2 + 2(\Psi_3)^3 = 0. \tag{68}$$

Write down the 12^{th} order coefficient of above equation, we get

$$-3\Psi_4^4\Psi_3^2\Psi_2^0 + 2(\Psi_3^2)^3 = 0. \tag{69}$$

In previous section, we have got

$$\begin{aligned}
\Psi_2^0 &= C_2, \\
\Psi_1^0 &= iC_1 \ {}_1Y_{1,0}, \\
\Psi_3^2 &= \frac{1}{2}\bar{\eth}^2\Psi_1^0, \\
\Psi_0^0 &= D^5 \ {}_2Y_{2,0}, \\
\Psi_3^3 &= -\frac{1}{2}\bar{\Psi}_1^0\Psi_2^0 - \frac{1}{6}\bar{\eth}^3\Psi_0^0, \\
\Psi_4^4 &= -\frac{1}{4}\bar{\eth}\Psi_3^3.
\end{aligned} \tag{70}$$

This constant is fixed in following way : from Komar integrals $M = \frac{1}{8\pi}\int_{S_\infty} *dt$ and $J = Ma = \frac{1}{16\pi}\int_{S_\infty} *d\phi$, we know $\Psi_2^0 = -M$, $\Psi_1^0 = 3iMa\sqrt{\frac{4\pi}{3}} \ {}_1Y_{1,0}$. Submit these into Eq.(70) then get

$$\begin{aligned}
\Psi_3^2 &= \frac{3iMa}{2}\sqrt{\frac{4\pi}{3}} \ {}_{-1}Y_{1,0} \quad , \\
\Psi_3^3 &= \left(\frac{3iM^2a}{2}\sqrt{\frac{4\pi}{3}} - \frac{1}{\sqrt{2}}D^5\right) \ {}_{-1}Y_{2,0} \quad , \\
\Psi_4^4 &= \left(-\frac{i\sqrt{6\pi}M^2a}{4} + \frac{D^5}{4}\right) \ {}_{-2}Y_{2,0} \quad .
\end{aligned} \tag{71}$$

Submit above result into Eq.(69), we find

$$0 = 3M\left(-\frac{i\sqrt{6\pi}M^2a}{4} + \frac{D^5}{4}\right) \ {}_{-2}Y_{2,0} + 2\left[\frac{3iMa}{2}\sqrt{\frac{4\pi}{3}} \ {}_{-1}Y_{1,0}\right]^2$$

$$= \left[3M\left(-\frac{i\sqrt{6\pi}M^2a}{4}+\frac{D^5}{4}\right)-\frac{3M^2a^2}{4}\sqrt{\frac{16\pi}{5}}\right]\ {}_{-2}Y_{2,0} \tag{72}$$

Solving the simple linear algebraic equation $\left[3M\left(-\frac{i\sqrt{6\pi}M^2a}{4}+\frac{D^5}{4}\right)-\frac{3M^2a^2}{4}\sqrt{\frac{16\pi}{5}}\right]=0$, we can fix the value of D^5 is

$$D^5 = Ma^2\sqrt{\frac{16\pi}{5}}+i\sqrt{6\pi}M^2a. \tag{73}$$

Submit above result into Eq.(60), it is easy to check the N-P constant of Kerr space-time is zero, which has been got by [19, 21].

In order to fix general D^k, we also use inductive method and fix them order by order. Suppose we have known D^k up to order n. To get the value of D^{n+1}, we consider the $(n+8)^{th}$ coefficient of Eq.(68), a long but direct calculation shows it should be

$$-3\Psi_4^{n+1}\Psi_3^2\Psi_2^0+\cdots=0, \tag{74}$$

here "$\cdots$" also mean terms which only contain lower order coefficients. From Eq.(63),(66),(67), we know

$$\begin{aligned}
\Psi_0^{n+1} &= \tilde{\Psi}_0^{n+1}+D^{n+1}\ {}_2Y_{n+3,0},\\
\Psi_1^{n+1} &= -\frac{1}{(n-3)}\bar{\eth}\Psi_0^{n+1}+\cdots.\\
\Psi_2^{n+1} &= -\frac{1}{n-2}\bar{\eth}\Psi_1^{n+1}+\cdots,\\
\Psi_3^{n+1} &= -\frac{1}{n-1}\bar{\eth}\Psi_2^{n+1}+\cdots,\\
\Psi_4^{n+1} &= -\frac{1}{n}\bar{\eth}\Psi_3^{n+1}+\cdots,
\end{aligned} \tag{75}$$

where $\tilde{\Psi}_0^{n+1}$ is the special solution of Eq.(64) which corresponds to Kerr solution. We have known Kerr solution belongs to Type-D, i.e. Eq.(74) holds for $\tilde{\Psi}_0^k$, so Eq.(74) can be written as

$$\frac{-3\Psi_3^2\Psi_2^0}{(n-3)(n-2)(n-1)n}\bar{\eth}^4\tilde{\Psi}_0^{n+1}+\cdots=0. \tag{76}$$

If the general Ψ_0^{n+1} in Eq.(75) also satisfies Eq.(74), i.e.

$$\frac{-3\Psi_3^2\Psi_2^0}{(n-3)(n-2)(n-1)n}\bar{\eth}^4\tilde{\Psi}_0^{n+1}+D^{n+1}\frac{-3\Psi_3^2\Psi_2^0}{(n-3)(n-2)(n-1)n}\bar{\eth}^4\ {}_2Y_{n+3,0}+\cdots=0. \tag{77}$$

Because terms in "$\cdots$" only contain lower order coefficients, they remain unchanged when we change $\tilde{\Psi}_0^{n+1}$ to the general Ψ_0^{n+1}. Obviously, $\frac{-3\Psi_3^2\Psi_2^0}{(n-3)(n-2)(n-1)n}\bar{\eth}^4\ {}_2Y_{n+3,0}$ is a non-zero function for any n, so the general solution of Ψ_0^{n+1} satisfies Eq.(74) means $D^{n+1}=0$ and $\Psi_0^{n+1}=\tilde{\Psi}_0^{n+1}$. This tells us that Kerr solution is the only solution which satisfies all requirements of our theorem.

Remark : in above subsection, we proved our theorem under the requirement that the whole space-time is type-D. In subsection 2.2, we have shown that the freedom of vacuum, stationary, axial-symmetric space-time are just a set of constants $\{D^k\}$. The reason why we need the condition type-D is to fix the value of $\{D^k\}$. If the type-D condition holds at several points in Bondi neighborhood and those points are not zero-points of $\frac{-3\Psi_3^2\Psi_2^0}{(n-3)(n-2)(n-1)n}\bar{\eth}^4\ {}_2Y_{n+3,0}$, then, it is easy to see that $\{D^k\}$ should be zero, i.e. type-D condition holds at several points will imply this condition holds in the whole neighborhood. This feature may be a practical method from the experiment prospective. That means only several points need to be checked for the type-D condition to see whether the space-time around us is Kerr space-time. This maybe a useful property for future gravitational experiments, such as "mapping space-time"[22].

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Appendix A

The asymptotic extension of Kerr space-time in B-S coordinates are

1) Null tetrad

$$\begin{aligned}
l^a &= \frac{\partial}{\partial r}, \\
n^a &= \frac{\partial}{\partial u} + \left[-\frac{1}{2} + \frac{M}{r} - \frac{Ma^2}{2r^3}(3\cos^2\theta - 1) + O(r^{-4})\right]\frac{\partial}{\partial r} \\
&\quad + \left[\frac{iMa}{2r^3}\cot\frac{\theta}{2} + O(r^{-4})\right]\frac{\partial}{\partial\zeta} + \left[-\frac{iMa}{2r^3}\cot\frac{\theta}{2} + O(r^{-4})\right]\frac{\partial}{\partial\bar{\zeta}}, \qquad (78)\\
m^a &= \left[-\frac{3iMa}{2\sqrt{2}r^2}\sin\theta + \frac{Ma^2}{\sqrt{2}r^3}\sin^2\theta + O(r^{-4})\right]\frac{\partial}{\partial r} \\
&\quad + O(r^{-4})\frac{\partial}{\partial\zeta} + \left[\frac{(1+\zeta\bar{\zeta})}{\sqrt{2}r} + O(r^{-4})\right]\frac{\partial}{\partial\bar{\zeta}}.
\end{aligned}$$

2) N-P coefficients

$$\begin{aligned}
\rho &= -\frac{1}{r} + O(r^{-5}), \\
\sigma &= -\frac{3Ma^2\sin\theta}{2r^4} + O(r^{-5}), \\
\alpha &= -\frac{\cot\theta}{2\sqrt{2}r} + \frac{3Ma^2\sin\theta\cos\theta}{2\sqrt{2}r^4} + O(r^{-5}), \\
\beta &= \frac{\cot\theta}{2\sqrt{2}r} - \frac{3iMa\sin\theta}{2\sqrt{2}r^3} + \frac{33Ma^2\sin\theta\cos\theta}{2\sqrt{2}r^4} + O(r^{-5}),
\end{aligned}$$

$$
\begin{aligned}
\tau &= -\frac{3iMa\sin\theta}{2\sqrt{2}r^3} + \frac{18Ma^2\sin\theta\cos\theta}{\sqrt{2}r^4} + O(r^{-5}),\\
\lambda &= -\frac{Ma^2\sin^2\theta}{4r^4} + O(r^{-5}),\\
\mu &= -\frac{1}{2r} + \frac{M}{r^2} + \frac{3iMa\cos\theta}{2r^3} - \frac{Ma^2(3\cos^2\theta-1)}{r^4} + O(r^{-5}),\\
\gamma &= \frac{M}{2r^2} + \frac{(2\sqrt{2}-1)3iMa\cos\theta}{\sqrt{2}r^3} - \frac{3Ma^2(3\cos^2\theta-1)}{4r^4} + O(r^{-5}),\\
\nu &= \frac{3iMa\sin\theta}{4\sqrt{2}r^3} - \frac{Ma^2\sin\theta\cos\theta}{\sqrt{2}r^4} + O(r^{-5}).
\end{aligned}
\tag{79}
$$

3) Weyl curvature

$$
\begin{aligned}
\Psi_0 &= \frac{3Ma^2\sin^2\theta}{r^5} + O(r^{-6}),\\
\Psi_1 &= \frac{3iMa\sin\theta}{\sqrt{2}r^4} - \frac{12Ma^2\sin\theta\cos\theta}{\sqrt{2}r^5} + O(r^{-6}),\\
\Psi_2 &= -\frac{M}{r^3} - \frac{3iMa\cos\theta}{4r^4} + \frac{3Ma^2(3\cos^2\theta-1)}{r^5} + O(r^{-6}),\\
\Psi_3 &= -\frac{3iMa\sin\theta}{2\sqrt{2}r^4} + \left[-\frac{3i}{2\sqrt{2}}M^2a\sin\theta + \frac{6i}{\sqrt{2}}Ma^2\sin\theta\cos\theta\right]r^{-5} + O(r^{-6}),\\
\Psi_4 &= \frac{3Ma^2\sin^2\theta}{4r^5} + O(r^{-6}).
\end{aligned}
\tag{80}
$$

Appendix B

Some spin-weight harmonics

$$
\begin{aligned}
Y_{0,0} &= \frac{1}{\sqrt{4\pi}};\\
{}_1Y_{1,1} &= \sqrt{\frac{3}{16\pi}}(\cos\theta+1)e^{i\phi},\\
{}_1Y_{1,0} &= \sqrt{\frac{3}{8\pi}}\sin\theta,\\
{}_1Y_{1,-1} &= \sqrt{\frac{3}{16\pi}}(1-\cos\theta)e^{-i\phi};\\
Y_{1,1} &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi},\\
Y_{1,0} &= \sqrt{\frac{3}{4\pi}}\cos\theta,\\
Y_{1,-1} &= \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi};\\
{}_{-1}Y_{1,1} &= \sqrt{\frac{3}{16\pi}}(1-\cos\theta)e^{i\phi},
\end{aligned}
$$

$$
\begin{aligned}
{}_{-1}Y_{1,0} &= -\sqrt{\frac{3}{8\pi}}\sin\theta, \\
{}_{-1}Y_{1,-1} &= \sqrt{\frac{3}{16\pi}}(1+\cos\theta)e^{-i\phi}; \\
{}_{2}Y_{2,2} &= 3\sqrt{\frac{5}{96\pi}}(1+\cos\theta)^2 e^{2i\phi}, \\
{}_{2}Y_{2,1} &= 3\sqrt{\frac{5}{24\pi}}\sin\theta(1+\cos\theta)e^{i\phi}, \\
{}_{2}Y_{2,0} &= 3\sqrt{\frac{5}{16\pi}}\sin^2\theta, \\
{}_{2}Y_{2,-1} &= 3\sqrt{\frac{5}{24\pi}}\sin\theta(1-\cos\theta)e^{-i\phi}, \\
{}_{2}Y_{2,-2} &= 3\sqrt{\frac{5}{96\pi}}(1-\cos\theta)^2 e^{-2i\phi}; \\
{}_{1}Y_{2,2} &= -3\sqrt{\frac{5}{24\pi}}\sin\theta(1+\cos\theta)e^{2i\phi}, \\
{}_{1}Y_{2,1} &= 3\sqrt{\frac{5}{24\pi}}(2\cos\theta-1)(1+\cos\theta)e^{i\phi}, \\
{}_{1}Y_{2,0} &= 3\sqrt{\frac{5}{4\pi}}\sin\theta\cos\theta, \\
{}_{1}Y_{2,-1} &= 3\sqrt{\frac{5}{24\pi}}(2\cos\theta+1)(1-\cos\theta)e^{-i\phi}, \\
{}_{1}Y_{2,-2} &= 3\sqrt{\frac{5}{24\pi}}\sin\theta(1-\cos\theta)e^{-2i\phi}; \\
Y_{2,2} &= 3\sqrt{\frac{5}{16\pi}}\sin^2\theta e^{2i\phi}, \\
Y_{2,1} &= -6\sqrt{\frac{5}{16\pi}}\sin\theta\cos\theta e^{i\phi}, \\
Y_{2,0} &= \sqrt{\frac{5}{24\pi}}(3\cos^2\theta-1), \\
Y_{2,-1} &= 6\sqrt{\frac{5}{16\pi}}\sin\theta\cos\theta e^{-i\phi}, \\
Y_{2,-2} &= 3\sqrt{\frac{5}{16\pi}}\sin^2\theta e^{-2i\phi}; \\
{}_{-1}Y_{2,2} &= -3\sqrt{\frac{5}{24\pi}}\sin\theta(1-\cos\theta)e^{2i\phi}, \\
{}_{-1}Y_{2,1} &= 3\sqrt{\frac{5}{24\pi}}(2\cos\theta+1)(1-\cos\theta)e^{i\phi}, \\
{}_{-1}Y_{2,0} &= -\sqrt{\frac{5}{4\pi}}\sin\theta\cos\theta,
\end{aligned}
$$

$$
\begin{aligned}
{}_{-1}Y_{2,-1} &= 3\sqrt{\frac{5}{24\pi}}(2\cos\theta - 1)(1+\cos\theta)e^{-i\phi}, \\
{}_{-1}Y_{2,-2} &= 3\sqrt{\frac{5}{24\pi}}\sin\theta(1+\cos\theta)e^{-2i\phi}; \\
{}_{-2}Y_{2,2} &= 3\sqrt{\frac{5}{96\pi}}(1-\cos\theta)^2 e^{2i\phi}, \\
{}_{-2}Y_{2,1} &= -3\sqrt{\frac{5}{24\pi}}\sin\theta(1-\cos\theta)e^{i\phi}, \\
{}_{-2}Y_{2,0} &= 3\sqrt{\frac{5}{16\pi}}\sin^2\theta, \\
{}_{-2}Y_{2,-1} &= -3\sqrt{\frac{5}{24\pi}}\sin\theta(1+\cos\theta)e^{-i\phi}, \\
{}_{-2}Y_{2,-2} &= 3\sqrt{\frac{5}{96\pi}}(1+\cos\theta)^2 e^{-2i\phi}.
\end{aligned}
$$

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